

A 3-dimensional bosonic topological insulator and its exotic electromagnetic response

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Recently, many new types of bosonic symmetry-protected topological phases, including bosonic topological insulators, were predicted using group cohomology theory. The bosonic topological insulators have both $U(1)$ symmetry (particle number conservation) and time-reversal symmetry, described by symmetry group $U(1) \rtimes Z_2^T$. In this paper, we propose a projective construction of three-dimensional correlated gapped bosonic state with $U(1) \rtimes Z_2^T$ symmetry. The gapped bosonic insulator is formed by eight kinds of charge-1 bosons. We show that, in our bosonic state, an *electromagnetic* monopole with a unit magnetic charge is fermionic while an *electromagnetic* dyon with a unit magnetic charge and a unit electric charge is bosonic. This indicates that the constructed bosonic state is a non-trivial bosonic topological insulator, since in a trivial bosonic Mott insulator, the monopole is bosonic while the dyon is fermionic. We also constructed a three-dimensional correlated gapless bosonic insulator with $U(1) \rtimes Z_2^T$ symmetry, that has two emergent gapless $U(1)$ gauge fields, and excitations with fractional gauge charges for both the emergent and electromagnetic gauge fields. Both bosonic insulators can have protected conducting surface states. The gapless boundary excitations of the gapless bosonic insulator can even be fermionic.

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Introduction - A quantum ground state of a many-boson system can be in a spontaneously-symmetry-breaking state, or a topologically ordered (TO) state.[1–3] A TO state is defined by the following features: ground state degeneracy in a topologically nontrivial closed manifold,[1–3] or emergent fermionic/anyonic excitations,[4, 5] or chiral gapless edge excitations.[6, 7] If, in addition to a topological order, the ground state also has a symmetry, such a state will be referred as a “symmetry-enriched topological (SET) phase”.

Recently, it was predicted that even if the bosonic ground state does not break any symmetry and has a trivial topological order, it can still be in a non-trivial phase called “bosonic symmetry-protected topological (SPT) phase”.[10–12] Since the bosonic SPT phases have only trivial topological orders, a systematic description/construction of those SPT phases were obtained via group cohomology theory.[10–12] Many new SPT phases were predicted/constructed with all possible symmetries and in any dimensions, including three non-trivial bosonic SPT phases with $U(1)$ symmetry (particle number conservation) and time-reversal symmetry in three dimensions. We will refer those phases as bosonic topological insulators (BTI).

To realize bosonic TO phases or SPT phases, the interaction is crucial, since without interaction, bosons always tend to condense trivially. This fact hinders the perturbation approach if we want to realize TO or SPT phases. One useful approach is via the exactly soluble models, as in the fusion category approach[8, 9] and the group cohomology approach.[10–12] Recently, many other approaches were proposed, which are based on field theory, topological invariants, critical theory of surface, topolog-

ical response theory etc.[13–21, 23–25] A quite effective approach for strongly interacting systems is the “projective construction”.[26–37] It has been recently realized that the projective construction is also helpful in constructing bosonic SPT/SET states.[22, 38–41]. In particular, Ref. 22 provides a projective construction of a 3D bosonic topological insulator, using bosonic partons.

Roughly speaking, in the projective construction, the bosonic operator is split into a product of parton operators. The different kinds of partons can individually form different meanfield ground states. The physical ground state of the boson system is realized by projecting the direct product of multiple meanfield ground states into the physical Hilbert space \mathcal{H}_{phys} , in which the multiple partons are glued together into a physical boson on each site. In terms of path integral formulation, such a gluing process is done by introducing internal gauge fields that couple to partons.

It is now well-known that three dimensional *non-interacting fermionic* topological insulators (TI)[42] are classified by Z_2 . The free fermionic TI state is protected by $U(1) \rtimes Z_2^T$ symmetry where $U(1)$ global symmetry is related to particle number conservation and Z_2^T is the time-reversal symmetry. The trivial and non-trivial TI states can be labeled by the θ angle in $F \wedge F$ term. $\theta = 0$ (π) corresponds to the trivial (non-trivial) phase. It is natural to ask whether there exists a bosonic version of TI? In 2010, Swingle *et.al.* provided an interesting projective construction approach in which the boson creation operator is split into a singlet pair of spin-1/2 fermions[43]. It is assumed that the fermions are described by a nontrivial TI meanfield Hamiltonian, which breaks the internal $SU(2)$ gauge symmetry down

to Z_2 . The resultant physical ground state is proved to be a topologically ordered phase supporting fractional theta angle and emergent Z_2 gauge theory. This bosonic insulator is called “fractional bosonic topological insulator” (f -BTI), following Maciejko *et.al.* which proposed a fermionic version[44].

The above bosonic topological insulator state has a non-trivial topological order, so that it is not a SPT but a SET state. In the present work, we shall consider a new projective construction of a correlated bosonic state with eight kinds of charge-1 bosons. We will show that such a state realizes a nontrivial SPT state protected by $U(1) \rtimes Z_2^T$, i.e. a nontrivial BTI state. Note that, some other constructions of 3-dimensional SPT states with the same symmetry are proposed.[15, 22, 45, 46] It is not totally clear if those different constructions give rise to the same or different BTI states.

Projective construction - We will use a system with eight kinds of charge-1 bosons to realize a BTI. Those bosons are described by two sets of boson operators, each of which has four components. We split the boson operators into four different spin-1/2 fermions:

$$\begin{aligned} (b_{11}, b_{12}, b_{13}, b_{14}) &= (f_{1\uparrow}f_{2\uparrow}, f_{1\uparrow}f_{2\downarrow}, f_{1\downarrow}f_{2\uparrow}, f_{1\downarrow}f_{2\downarrow})^T, \\ (b_{21}, b_{22}, b_{23}, b_{24}) &= (f_{3\uparrow}f_{4\uparrow}, f_{3\uparrow}f_{4\downarrow}, f_{3\downarrow}f_{4\uparrow}, f_{3\downarrow}f_{4\downarrow})^T. \end{aligned} \quad (1)$$

Under time reversal, the above fermions transform as the usual spin-1/2 fermions, but with an additional exchange $(f_1, f_3) \leftrightarrow (f_2, f_4)$, and the bosons transform as

$$\begin{aligned} b_{11} &\rightarrow -b_{14}, b_{12} \rightarrow b_{12}, b_{13} \rightarrow b_{13}, b_{14} \rightarrow -b_{11}, \\ b_{21} &\rightarrow -b_{24}, b_{22} \rightarrow b_{22}, b_{23} \rightarrow b_{23}, b_{24} \rightarrow -b_{21} \end{aligned} \quad (2)$$

For instance, $b_{12} = f_{1\uparrow}f_{2\downarrow} \rightarrow -f_{1\downarrow}f_{2\uparrow} \rightarrow -f_{2\downarrow}f_{1\uparrow} = f_{1\uparrow}f_{2\downarrow} = b_{12}$ such that b_{12} is unchanged, where the first arrow is $f_{\downarrow} \rightarrow -f_{\uparrow}, f_{\uparrow} \rightarrow f_{\downarrow}$ and the second arrow is exchange of labels: $1 \leftrightarrow 2$. In this projective construction, there are two independent internal $U(1)$ gauge fields, a_μ^1 and a_μ^2 , couple to f_1, f_2 and f_3, f_4 , respectively. The assignment of gauge charges carried by f_s ($s = 1, 2, 3, 4$) is following. f_1 and f_3 carry +1 gauge charges of a^1 and a^2 respectively; f_2 and f_4 carry -1 gauge charges of a_μ^1 and a_μ^2 respectively. Each boson carries +1 fundamental electric charge of EM field A_μ such that one can make the following assignment for fermions: $f_{1,2,3,4}$ carry $\alpha, 1 - \alpha, \alpha, 1 - \alpha$ electric charge of A_μ correspondingly. α is integer-valued, so that all the gauge charges are quantized as integers.

The projective construction is a natural way to obtain TO states. To obtain SPT states, we must prohibit the emergence of topological order, by *at least* considering the confined phase of internal gauge fields, where the dyons of the internal gauge fields play a very important role. A generic dyon carries gauge charges and magnetic charges of all the gauge fields (including internal gauge fields and

external electromagnetic (EM) field). Thus a dyon is labeled by a set of quantum numbers that describe those gauge charges and magnetic charges. Specially, a dyon which doesn't carry net gauge charge is called a monopole in our definition.

Different choices of dyonic condensates generally lead to different phases. In this paper, we consider a state with two dyon condensations, which is designed to respects the $U(1) \rtimes Z_2^T$ symmetry. The two dyon condensations completely break the internal gauge symmetry which leads to a state with trivial topological order. We show that such a trivial TO state supports a bosonic EM-dyon (carrying one unit magnetic charge and one unit of electric charge of EM field) and a fermionic EM-monopole (merely carrying one unit magnetic charge of EM field). Such an exotic EM response can not be realized in vacuum (or trivial bosonic Mott insulators) and in fermionic TI. In vacuum, the EM-dyon and EM-monopole are fermionic and bosonic, respectively, while in nontrivial TI the EM-dyon and EM-monopole don't exist as the electric charge is always nonzero half-integer when magnetic charge is one unit.

Algebraic bosonic insulator (ABI)- Before discussing the above BTI, let us first use the projective construction to study a gapless bosonic insulator (without the dyon condensation) which is called “Algebraic bosonic insulator (ABI)”. ABI is obtained from a meanfield state of fermionic partons, where the four spin-1/2 fermions form four *identical* non-trivial TI states. The corresponding physical bosonic state is obtained from the mean-field fermionic parton state by projecting into the physical Hilbert space of the bosonic states. Here, we assume that we can design a bosonic Hamiltonian to realize such a bosonic state. In this paper, we just want to study the physical properties of the constructed bosonic state.

First, the bosonic state does not break the $U(1) \rtimes Z_2^T$ symmetry but the bulk is gapless since it contains two emergent gapless $U(1)$ gauge bosons described by a_μ^1 and a_μ^2 . Such an ABI is *not* a SPT state which is always gapped. In addition to the emergent gapless $U(1)$ gauge bosons, ABI also contains many dyon excitations, which may carry fractional charges and emergent Fermi statistics. To describe those dyon excitations systematically, let us assume that each fermion (f_s) couples to its own gauge field ($A_\mu^{f_s}$) with “+1” gauge charge. In fact, $A_\mu^{f_s}$ are combinations of a_μ^1, a_μ^2 and A_μ :

$$\begin{aligned} A_\mu^{f_1} &\equiv a_\mu^1 + \alpha A_\mu, A_\mu^{f_2} \equiv -a_\mu^1 + (1 - \alpha)A_\mu, \\ A_\mu^{f_3} &\equiv a_\mu^2 + \alpha A_\mu, A_\mu^{f_4} \equiv -a_\mu^2 + (1 - \alpha)A_\mu. \end{aligned} \quad (3)$$

A dyon can carry the magnetic charges for each of the four gauge fields (A^{f_s}), which are labeled by $\{N_m^{(s)}\} \in \mathbb{Z}$ ($s = 1, \dots, 4$). However, the four magnetic charges $\{N_m^{(s)}\}$ are not independent. They are given by the three magnetic charges of a_μ^1, a_μ^2 and A_μ , denoted as $N_m^{a_1}, N_m^{a_2}$

TABLE I: Quantum numbers of four kinds of dyons.

Dyonic fields	N_m^{a1}	N_m^{a2}	N_M	$N_f^{(1)}$	$N_f^{(2)}$	$N_f^{(3)}$	$N_f^{(4)}$	Sgn	$N_m^{(1)}$	$N_m^{(2)}$	$N_m^{(3)}$	$N_m^{(4)}$	N_A	N^{a1}	N^{a2}
ϕ_1	1	0	0	1/2	1/2	0	0	+	1	-1	0	0	1/2	0	0
ϕ_2	0	1	0	0	0	1/2	1/2	+	0	0	1	-1	1/2	0	0
ϕ_3	0	0	1	1/2	0	1/2	0	+	1	0	1	0	1	1/2	1/2
ϕ_4	0	0	1	1/2	0	-1/2	-1	-	1	0	1	0	0	1/2	1/2

and N_M :

$$\begin{aligned} N_m^{(1)} &= N_m^{a1} + \alpha N_M, N_m^{(2)} = -N_m^{a1} + (1 - \alpha) N_M, \\ N_m^{(3)} &= N_m^{a2} + \alpha N_M, N_m^{(4)} = -N_m^{a2} + (1 - \alpha) N_M. \end{aligned} \quad (4)$$

Thus, it is better to label magnetic sector using $(N_m^{a1}, N_m^{a2}, N_M)$.

A dyon also carries the fermion numbers of f_1, f_2, f_3, f_4 , which are labeled by $\{N_f^{(s)}\}$, $s = 1, \dots, 4$. They are related to magnetic charges in the following way: $N_f^{(s)} = n_f^s + \frac{1}{2} N_m^{(s)}$ where the second terms are polarization charge clouds due to Witten effect[47, 48] and $n_f^{(s)}$ are integer-valued, indicating that integer numbers of fermions are attached to the dyon. A useful observation is that $N_f^{(s)}$ is an integer if $N_m^{(s)}$ is even and $N_f^{(s)}$ is a half-integer if $N_m^{(s)}$ is odd. The four fermion numbers $\{N_f^{(s)}\}$ are independent labels despite that $\{N_m^{(s)}\}$ are not independent. In summary, our constructed bosonic state can have many kinds of dyonic excitations which are labeled by 7 *independent* quantum numbers, i.e. $(N_m^{a1}, N_m^{a2}, N_M, N_f^{(1)}, N_f^{(2)}, N_f^{(3)}, N_f^{(4)})$.

Quantum Statistics and “mutual statistics”- A generic dyon can be viewed as $N_m^{(s)}$ magnetic charges of A_μ^{fs} gauge field attached by $n_f^{(s)}$ f_s fermions. The quantum statistics of such a dyon is given by

$$Sgn = \prod_s (-1)^{N_m^{(s)} n_f^{(s)}} (-1)^{n_f^{(s)}}, \quad (5)$$

where $+/-$ represents bosonic/fermionic.[49] The first part, $(-1)^{N_m^{(s)} n_f^{(s)}}$, is due to the interaction between the magnetic charge $N_m^{(s)}$ and the gauge charge $n_f^{(s)}$ of the dyon, and the second part, $(-1)^{n_f^{(s)}}$, is due to the Fermi statistics from the attachment of $n_f^{(s)}$ f_s fermions.

Another important property of dyons is their 3D “mutual statistics”, which is defined below. Two dyons with different quantum numbers may perceive a nonzero quantum Berry phase mutually. More specifically, let’s fix one dyon (“D1”) at origin and move another dyon (“D2”) (labeled by symbol with primes) along a closed trajectory which forms a solid angle Ω with respect to the origin. Under this circumstance, one can calculate the Berry phase Φ_{Berry} that is added into D2’s wavefunction:

$$\Phi_{\text{Berry}} = \frac{1}{2} \left[\sum_s N_m^{(s)} N_f^{(s)'} - \sum_s N_m^{(s)'} N_f^{(s)} \right] \Omega. \quad (6)$$

If $\Phi_{\text{Berry}} \neq 0$ for any Ω , i.e. $\sum_s N_m^{(s)} N_f^{(s)'} \neq \sum_s N_m^{(s)'} N_f^{(s)}$, these two dyons then have a non-trivial “mutual statistics”.

In Table I, we list properties of four selected dyons ϕ_1, ϕ_2, ϕ_3 , and ϕ_4 . We have assumed $\alpha = 1$ such that the electric charge carried by a dyon is given by: $N_A = N_f^{(1)} + N_f^{(3)}$. Also, the a_μ^1 and a_μ^2 gauge charges are given by $N^{a1} = N_f^{(1)} - N_f^{(2)}$ and $N^{a2} = N_f^{(3)} - N_f^{(4)}$.

The two dyons ϕ_1 and ϕ_2 are bosonic and have trivial “mutual statistics”. The dyons ϕ_3 and ϕ_4 also have trivial “mutual statistics” with dyons ϕ_1 and ϕ_2 . However, the dyon ϕ_3 has a non-trivial “mutual statistics” with the dyon ϕ_4 . Also the dyon ϕ_3 is a boson and the dyon ϕ_4 is a fermion. We also note that all the four dyons carry fractional gauge charges. The ABI is indeed a very interesting state.

At mean-field level, since each of f_s fermions forms a non-trivial TI, those fermions will produce gapless surface states described by four Dirac fermions. Beyond mean-field theory, those gapless surface Dirac fermions (possibly with Fermi energies away from the nodes) will interact with the two emergent $U(1)$ gauge fields. Such an interaction may make the surface state a Marginal-Fermi liquid.[50, 51]

Dyon condensation with $U(1) \times \mathbb{Z}_2^T$ symmetry - In the rest of this paper, we will consider how to realize a less interesting *BTI*, without emergent gapless $U(1)$ gauge bosons. First, we note that the two internal $U(1)$ gauge fields a_μ^1 and a_μ^2 have strong quantum fluctuations, and their “fine structure constants” are of order 1. The ABI discussed above is realized by the Coulomb phase of the two internal $U(1)$ gauge fields.

However, it is also possible (depending on the physical boson Hamiltonian) that the two internal $U(1)$ gauge fields are in a confined phase due to too strong quantum fluctuations. Due to the strong quantum fluctuations, the internal $U(1)$ gauge field configuration will contain many monopoles and even more general dyons. In this case, the confined phases can be viewed as *dyon condensed phases*. [21, 22] So, in the following, we will consider confined phases via dyon condensations.

As we have seen that our ABI has many kinds of dyons. Depending on the details of the physical boson Hamiltonian, one or a few of those dyons may condense. Different dyon condensations will lead to many different phases, such as superfluid phases, time-reversal symme-

try breaking phases, insulating phases, etc. In this paper, we will look for dyon condensations that do not break the $U(1) \rtimes Z_2^T$ symmetry.

We note that under the time reversal Z_2^T transformation (2), the a_μ^1 and a_μ^2 gauge charges change sign, due to the additional exchange $(f_1, f_3) \leftrightarrow (f_2, f_4)$. In this case, a dyon transforms as (when $N_M = 0$)

$$\begin{aligned} N_m^{a1} &\rightarrow N_m^{a1}, \quad N_m^{a2} \rightarrow N_m^{a2}, \quad N_f^{(1)} \rightarrow N_f^{(2)}, \quad N_f^{(2)} \rightarrow N_f^{(1)}, \\ N_f^{(3)} &\rightarrow N_f^{(4)}, \quad N_f^{(4)} \rightarrow N_f^{(3)}. \end{aligned} \quad (7)$$

For instance, $N_m^{a1} \rightarrow -N_m^{a1} \rightarrow N_m^{a1}$ where the first arrow is usual time reversal transformation of magnetic charge, and the second arrow is due to the exchange $1 \leftrightarrow 2$ noting that $f_{1,2}$ carry opposite gauge charges $(+1, -1)$ of a^1 . To obtain a time reversal symmetric state that breaks both the internal $U(1)$ gauge symmetry, we consider condensations of two special dyons (ϕ_1, ϕ_2) in Table I. We note that ϕ_1 and ϕ_2 are bosons with trivial “mutual statistics”. Thus they can condense together. (Only bosonic dyons with trivial “mutual statistics” can condense together.) The two dyons ϕ_1 and ϕ_2 are symmetric under Z_2^T . Thus their condensations do not break the time-reversal symmetry. But will such a dyon condensed state respect the $U(1)$ symmetry and behave like an insulator?

To answer such a question, let us write down the effective Lagrangian of the ϕ_1, ϕ_2 dyons in real time ($A_0 = 0$):

$$\begin{aligned} \mathcal{L} = & \phi_1^* i \partial_t \phi_1 + \frac{1}{2m} |(-i\nabla + \tilde{\mathbf{a}}^1 + \frac{1}{2}\mathbf{A})\phi_1|^2 \\ & + \phi_2^* i \partial_t \phi_2 + \frac{1}{2m} |(-i\nabla + \tilde{\mathbf{a}}^2 + \frac{1}{2}\mathbf{A})\phi_2|^2 - V(\phi_1, \phi_2), \end{aligned} \quad (8)$$

where, $\frac{1}{2}\mathbf{A}$ is present since both of ϕ_1, ϕ_2 carry $1/2$ electric charge of EM field. Note that in this paper, we treat \mathbf{A} as a non-dynamical probe field. ϕ_1, ϕ_2 also carry magnetic charges of a_μ^1, a_μ^2 . To describe such couplings, one needs to apply the *electric-magnetic duality* transformation for a_μ^1 and a_μ^2 . In the new basis, both ϕ_1 and ϕ_2 carry zero magnetic charges, but nonzero gauge charges ($N^{\tilde{a}1} = 1, N^{\tilde{a}2} = 0$ for ϕ_1 ; $N^{\tilde{a}1} = 0, N^{\tilde{a}2} = 1$ for ϕ_2). This leads to the $\tilde{\mathbf{a}}^1$ and $\tilde{\mathbf{a}}^2$ terms in the above effective Lagrangian, where $\tilde{\mathbf{a}}^1$ and $\tilde{\mathbf{a}}^2$ are the two dual gauge potentials.

In the dyon condensed state $\phi_1, \phi_2 \neq 0$, both the internal gauge fields are gapped and satisfy [the last two equations are dual to the first two under the duality relation $(\mathbf{E}, \mathbf{B}) \rightarrow (\mathbf{B}, -\mathbf{E})$ or $(\nabla \times \tilde{\mathbf{A}}, \nabla \times \mathbf{A}) \rightarrow (\nabla \times \mathbf{A}, -\nabla \times \tilde{\mathbf{A}})]$

$$\tilde{\mathbf{a}}^1 = -\frac{\mathbf{A}}{2}, \quad \tilde{\mathbf{a}}^2 = -\frac{\mathbf{A}}{2}, \quad \mathbf{a}^1 = \frac{\tilde{\mathbf{A}}}{2}, \quad \mathbf{a}^2 = \frac{\tilde{\mathbf{A}}}{2} \quad (9)$$

which indicates that internal gauge fields can not fluctuate. We see that the dyon condensations do not generate the \mathbf{A}^2 term. Thus the dyon condensed state also respects the $U(1)$ symmetry and represents a fully gapped insulator. Also since the dyons carry unit charge of the

(dual) internal gauge fields, the two $U(1)$ gauge symmetries are fully broken. This suggests that the gapped insulator has a trivial topological order.

To probe the dyon condensed state, let us consider the ϕ_3, ϕ_4 dyons in the ϕ_1, ϕ_2 condensed state. Since the ϕ_3, ϕ_4 dyons have trivial “mutual statistics” with the ϕ_1, ϕ_2 dyons, ϕ_3, ϕ_4 are not confined and appear as finite energy excitations in the ϕ_1, ϕ_2 condensed state. We also note that the f_s fermions all have non-trivial “mutual statistics” with the ϕ_1, ϕ_2 dyons, and thus those fermionic excitations are confined. The surface fermionic gapless excitations described by four f_s Dirac fermions are also confined by the ϕ_1, ϕ_2 condensation. The confinement behaves like a strong attraction between f_1 and f_2 fermions, as well as between f_3 and f_4 fermions, which may make the surface into a superconducting state.

To calculate quantum numbers of the ϕ_3, ϕ_4 dyons, we start with the effective Lagrangian of the ϕ_3, ϕ_4 dyons:

$$\begin{aligned} \mathcal{L} = & \phi_3^* i \partial_t \phi_3 + \frac{1}{2m_3} |(-i\nabla + \frac{1}{2}\mathbf{a}^1 + \frac{1}{2}\mathbf{a}^2 + \mathbf{A} + \tilde{\mathbf{A}})\phi_3|^2 \\ & + \phi_4^* i \partial_t \phi_4 + \frac{1}{2m_4} |(-i\nabla + \frac{1}{2}\mathbf{a}^1 + \frac{1}{2}\mathbf{a}^2 + \tilde{\mathbf{A}})\phi_4|^2. \end{aligned} \quad (10)$$

The coupling to the dual EM gauge potential \tilde{A}_μ is due to the non-zero N_M (the EM magnetic charge) carried by ϕ_3 and ϕ_4 dyons. Since $\mathbf{a}^1, \mathbf{a}^2$ are tied to \mathbf{A} in the ϕ_1, ϕ_2 condensed state, we must replace $\mathbf{a}^1, \mathbf{a}^2$ in Eq. (10) by \mathbf{A} using Eq. (9). Then, from their coupling to the EM gauge potential \mathbf{A} , we obtain the net EM electric charge N_A^{net} of $\phi_{3,4}$. We see that ϕ_3 is an EM dyon with a unit electric charge and a unit magnetic charge, while ϕ_4 is an EM monopole with only a unit magnetic charge.

We find that, in the ϕ_1, ϕ_2 dyon condensed state, the EM dyon is bosonic instead of fermionic (as in vacuum), while the EM monopole is fermionic instead of bosonic (as in vacuum). According to Ref. 21, such properties imply that the dyon condensed state corresponds to a nontrivial BTI.

Similar to ϕ_3, ϕ_4 , we can consider a general dyon Φ labeled by $(N_m^{a1}, N_m^{a2}, N_M, N_f^{(1)}, N_f^{(2)}, N_f^{(3)}, N_f^{(4)})$ whose net electric charge $N_A^{net} = N_A - \frac{1}{2}(N_m^{a1} + N_m^{a2})$. We also require that $\frac{1}{2}N_M = N_f^{(1)} - N_f^{(2)} = N_f^{(3)} - N_f^{(4)}$ to trivialize mutual statistics between Φ and $\phi_{1,2}$ such that Φ is deconfined. We find that all deconfined dyon excitations with zero EM magnetic charges ($N_M = 0$) are bosons. This is consistent with the fact that the ϕ_1, ϕ_2 condensed state has a trivial topological order.

Conclusions. - We have constructed a gapless bosonic insulator (ABI) and a gapped bosonic topological insulator (BTI) with $U(1) \rtimes Z_2^T$ symmetry in three dimensions. The ABI state has two emergent $U(1)$ gauge fields and emergent fractional gauge charges. It can also have conducting surface states, which are described by gapless fermionic excitations. Then, we show that one particular confined phase of the two emergent $U(1)$ gauge fields

(described by a condensation of two dyons) gives rise to a non-trivial SPT state – a BTI with trivial topological order. We probe the BTI by adding an electromagnetic dyon and an electromagnetic monopole,[21] to investigate the non-trivial SPT order in the bulk. BTI can also have conducting surface states at the bulk boundary. However, the gapless fermionic boundary excitations at the mean-field level are confined by the dyon condensations.

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- [1] X.-G. Wen, Phys. Rev. B **40**, 7387 (1989).
 - [2] X.-G. Wen and Q. Niu, Phys. Rev. B **41**, 9377 (1990).
 - [3] X.-G. Wen, Int. J. Mod. Phys. B **4**, 239 (1990).
 - [4] B. I. Halperin, Phys. Rev. Lett., **52**, 1583 (1984).
 - [5] D. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984)
 - [6] B. I. Halperin, Phys. Rev. B **25**, 2185 (1982)
 - [7] X.-G. Wen, Int. J. Mod. Phys. B **6**, 1711 (1992)
 - [8] M. Freedman, C. Nayak, K. Shtengel, K. Walker, and Z. Wang, Ann. Phys. (NY) **310**, 428 (2004)
 - [9] M. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005), <http://arxiv.org/abs/cond-mat/0404617cond-mat/0404617>
 - [10] Xie Chen, Z. C. Gu, Z. X. Liu, and X. G. Wen, arXiv:1106.4772.
 - [11] Xie Chen, Z. C. Gu, and X. G. Wen, Phys. Rev. B **82**, 155138 (2010).
 - [12] X. Chen, Z.-C. Gu, Z.-X. Liu, X.-G. Wen, Science **338**, 1604 (2012); X. L. Qi, Science **338**, 1550 (2012).
 - [13] M. Levin and A. Stern, Phys. Rev. Lett. **103**, 196803 (2009), arXiv:0906.2769.
 - [14] Y.-M. Lu and A. Vishwanath, Phys. Rev. **86**, 125119 (2012), arXiv:1205.3156.
 - [15] Ashvin Vishwanath, T. Senthil, Phys. Rev. X **3**, 011016 (2013), arXiv:1209.3058;
 - [16] F. J. Burnell, X. Chen, L. Fidkowski, and A. Vishwanath (2013), arXiv:1302.7072.
 - [17] Meng Cheng and Zheng-Cheng Gu (2013), arXiv:1302.4802.
 - [18] Xie Chen, Fa Wang, Y.-M. Lu, and D.-H. Lee (2013), arXiv:1302.3121.
 - [19] X.-G. Wen (2013), arXiv: 1301.7675.
 - [20] Cenke Xu (2013), arXiv: 1301.6172.
 - [21] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher, arXiv: 1302.6535 (2013).
 - [22] M. A. Metlitski, C. L. Kane, and M. P. A. Fisher, to appear. See video on-line <http://www.birs.ca/events/2013/5-day-workshops/>
 - [23] Ling-Yan Hung and X.-G. Wen (2013), arXiv:1211.2767.
 - [24] T. Senthil and M. Levin (2013). arXiv:1206.1604.
 - [25] Cenke Xu (2012), arXiv:1209.4399.
 - [26] G. Baskaran, Z. Zou, and P. W. Anderson, Solid State Comm. **63**, 973 (1987).
 - [27] G. Baskaran and P. W. Anderson, Phys. Rev. B **37**, 580 (1988).
 - [28] I. Affleck and J. B. Marston, Phys. Rev. B **37**, 3774 (1988).
 - [29] G. Kotliar and J. Liu, Phys. Rev. B **38**, 5142 (1988).
 - [30] Y. Suzumura, Y. Hasegawa, and H. Fukuyama, J. Phys. Soc. Jpn. **57**, 2768 (1988).
 - [31] I. Affleck, Z. Zou, T. Hsu, and P. W. Anderson, Phys. Rev. B **38**, 745 (1988).
 - [32] E. Dagotto, E. Fradkin, and A. Moreo, Phys. Rev. B **38**, 2926 (1988).
 - [33] X.-G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11413 (1989).
 - [34] X.-G. Wen, Phys. Rev. B **44**, 2664 (1991).
 - [35] P. A. Lee and N. Nagaosa, Phys. Rev. B **45**, 5621 (1992).
 - [36] C. Mudry and E. Fradkin, Phys. Rev. B **49**, 5200 (1994).
 - [37] X.-G. Wen and P. A. Lee, Phys. Rev. Lett. **76**, 503 (1996), cond-mat/9506065.
 - [38] T. Grover and A. Vishwanath (2012), arXiv:1210.0907.
 - [39] Y.-M. Lu and D.-H. Lee (2012), arXiv:1210.0909.
 - [40] Y.-M. Lu and D.-H. Lee (2012), arXiv:1212.0863.
 - [41] Peng Ye and Xiao-Gang Wen, arXiv:1212.2121 (2012).
 - [42] J. E. Moore and L. Balents, Phys. Rev. B **75**, 121306 (2007); L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. **98**, 106803 (2007); R. Roy, Phys. Rev. B, **79**, 195322 (2009); M. Z. Hasan, C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010); J. E. Moore, Nature **464**, 194 (2010); X.-L. Qi, S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
 - [43] B. Swingle, M. Barkeshli, J. McGreevy, and T. Senthil, Phys. Rev. B **83**, 195139 (2011), arXiv:1005.1076 (2010).
 - [44] J. Maciejko, X. L. Qi, A. Karch, S. C. Zhang, Phys. Rev. Lett. **105**, 246809 (2010).
 - [45] Cenke Xu and T. Senthil, (2013). arXiv:1301.6172.
 - [46] C. Wang and T. Senthil, (2013). arXiv:1302.6234.
 - [47] E. Witten, Phys. Lett. B **86**, 283 (1979).
 - [48] G. Rosenberg and M. Franz, Phys. Rev. B **82**, 035105 (2010).
 - [49] A. S. Goldhaber, R. MacKenzie, and F. Wilczek, Mod. Phys. Lett. A, **4**, 21 (1989).
 - [50] William Witczak-Krempa, Ting Pong Choy, Yong Baek Kim, Phys. Rev. B, **82**, 165122 (2010).
 - [51] B. L. Altshuler, L. B. Ioffe, and A. J. Millis, Phys. Rev. B, **50**, 14048 (1994); J. Polchinski, Nucl. Phys. B, **422**, 617 (1994) C. Nayak, and F. Wilczek, Nucl. Phys. B, **417**, 359 (1994); Nucl. Phys. B, **430**, 534 (1994); Y. B. Kim, P. A. Lee, and Xiao-Gang Wen, Phys. Rev. B **52**, 17275 (1995).